Int J Theor Phys (2008) 47: 1016–1020
DOI 10.1007/s10773-007-9529-5

Axially Symmetric Inflationary Universe in General Relativity

D.R.K. Reddy · R.L. Naidu · S. Atchuta Rao

Received: 6 June 2007 / Accepted: 17 August 2007 / Published online: 29 August 2007

Abstract We have investigated a simple axially symmetric inflationary universe in the presence of mass less scalar field with a flat potential. To get an inflationary universe, we have considered a flat region in which potential V is constant. Some physical properties of the universe are also discussed.

Keywords Inflationary universe · General relativity

1 Introduction

Scalar fields are the simplest classical fields and there exist an extensive literature containing numerous solutions of the Einstein equations where the scalar field is minimally coupled to the gravitational field. In particular, self-interacting scalar fields play a central role in the study of inflationary cosmology. Many authors [1–5] studied different aspects of scalar field in the evolution of the universe and FRW models.

Inflationary universes play a vital role in solving a number of outstanding problems in cosmology like the homogeneity, the isotropy and the flatness of the observed universe. Guth [6], Linde [7] and La and Steinhardt [8] are some of the authors who have investigated several aspects of the inflationary universes in general relativity. Using the concept of Higgs field ϕ with potential $V(\phi)$ has a flat region and the ϕ field evolves slowly but the universe expands in an exponential way due to vacuum field energy [9]. It is assumed that the scalar field will take sufficient time to cross the flat region so that the universe expands sufficiently

D.R.K. Reddy (✉)
MVGR College of Engg, Vizianagaram, India
e-mail: reddy_einstein@yahoo.com

R.L. Naidu
Department of Mathematics, GMR Institute of Technology, Rajam, India

S.A. Rao
Department of Statistics, Andhra University, Visakhapatnam, India

to become homogeneous and isotropic on the scale of the order of the horizon size. Bhattacharjee and Baruah [10], Raj Bali and Jain [11] and Rahaman et al. [12] have, recently, studied the role of self-interacting scalar fields in inflationary cosmology.

In this paper, we have investigated an axially symmetric inflationary cosmological model in the presence of massless scalar field with a flat potential in general relativity. To get a determinate solution, we have considered a flat region in which potential is constant.

2 Metric and Field Equations

We consider the axially symmetric metric in the form [13]

$$ds^2 = dt^2 - A^2(t)[d\chi^2 + f^2(\chi)d\psi^2] - B^2(t)dz^2 \tag{1}$$

with the convention $x^1 = \chi$, $x^2 = \psi$, $x^3 = Z$ and $x^4 = t$ and A and B are functions of the proper time t alone and f is function of the coordinate χ alone.

In this case of gravity minimally coupled to a scalar field $V(\phi)$ the Lagrangian is [9]

$$L = \int \left[R - \frac{1}{2} g^{ij} \phi_{,i} \phi_{,j} - V(\phi) \right] \sqrt{-g}\, d^4x \tag{2}$$

which on variations of L with respect to the dynamical fields leads to Einsein field equations.

$$R_{ij} - \frac{1}{2} g_{ij} R = -T_{ij} \tag{3}$$

with

$$T_{ij} = \phi_{,i}\phi_{,j} - \left[\frac{1}{2}\phi_{,k}\phi^{,k} + V(\phi) \right] g_{ij}, \tag{4}$$

$$\phi^{i}_{;i} = -\frac{dV}{d\phi} \tag{5}$$

where comma and semi colon indicate ordinary and covariant differentiation respectively. Notations have their usual meaning and units are taken so that $8\pi G = c = 1$.

Now the Einstein field equations (3) for the metric (1) are given by

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} + \frac{\phi_4^2}{2} + V(\phi) = 0, \tag{6}$$

$$2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \frac{f_{11}}{A^2 f} + \frac{\phi_4^2}{2} + V(\phi) = 0, \tag{7}$$

$$\frac{f_{11}}{A^2 f} - \frac{A_4^2}{A^2} - 2\frac{A_4 B_4}{AB} + \frac{\phi_4^2}{2} - V(\phi) = 0 \tag{8}$$

and (5) for the scalar field takes the form

$$\phi_{44} + \phi_4 \left(\frac{2A_4}{A} + \frac{B_4}{B} \right) + \frac{dV}{d\phi} = 0 \tag{9}$$

where suffixes 1 and 4 after a field variable represent differentiation with respect to χ and t respectively.

3 Solutions and the Model

The functional dependence of the metric together with (8) imply that [13]

$$\left.\begin{aligned} &\frac{f_{11}}{f}=k^2, \quad k=\text{Constant}\\ &\text{If } k=0, \quad \text{then} \quad f(\chi)=(\text{Constant})\chi, \quad 0<\chi<\infty \end{aligned}\right\}. \tag{10}$$

This constant can be made to 1 by choosing units for ψ. Thus we shall have

$$f(\chi)=\chi \tag{11}$$

resulting in the flat model of the universe [14].

Also, since we are interested in inflationary solutions, the flat region is considered where potential is constant [9], i.e.,

$$V(\phi)=\text{Constant}=V_0(\text{say}). \tag{12}$$

Now the field equations (6–9), with the help of (11) and (12), take the form

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}+\frac{\phi_4^2}{2}-V_0=0, \tag{13}$$

$$2\frac{A_{44}}{A}+\frac{A_4^2}{A^2}+\frac{\phi_4^2}{2}+V_0=0, \tag{14}$$

$$\frac{A_4^2}{A^2}+2\frac{A_4B_4}{AB}-\frac{\phi_4^2}{2}+V_0=0, \tag{15}$$

$$\phi_{44}+\phi_4\left(\frac{2A_4}{A}+\frac{B_4}{B}\right)=0. \tag{16}$$

The field equations (13–16) reduce to

$$\begin{aligned} &\frac{A_{44}}{A}-\frac{B_{44}}{B}+\frac{A_4^2}{A^2}-\frac{A_4B_4}{AB}=0,\\ &A^2B\phi=k_1 \end{aligned} \tag{17}$$

where k_1 is a constant of integration.

Here, we also assume the relation between metric coefficients, i.e., $A=\mu B^n$, because of the fact that the field equations are highly non-linear. Using this relation the field equations (17) admit the exact solution.

$$\left.\begin{aligned} &A=\mu[(n+1)(k_1t+k_2)]^{\frac{n}{n+1}}\\ &B=[(n+1)(k_1t+k_2)]^{\frac{1}{n+1}}\\ &\phi=\left(\frac{\phi_0}{\mu^2}\right)[(n+1)(k_1t+k_2)]^{-(\frac{2n+1}{n+1})} \end{aligned}\right\} \tag{18}$$

where k_1, k_2 and ϕ_0 are constants of integration. After a suitable choice of coordinates and constants of integration, the inflationary model corresponding to the solution (18) can be written as

$$ds^2 = dT^2 - \mu^2[(n+1)T]^{\frac{2n}{n+1}}[d\chi^2 + \chi^2 d\psi^2] - [(n+1)T]^{\frac{2}{n+1}} dz^2. \tag{19}$$

4 Some Physical Properties

The model (19) represents an exact, axially symmetric inflationary cosmological model in Einstein's theory of gravitation. The model has no initial singularity, i.e., at $T = 0$. This model is simple and elegant when compared to the inflationary model obtained by Raj Bali and Jain [11].

The scalar field and the potential in the model are

$$\phi = \left(\frac{\phi_0}{\mu^2}\right)[(n+1)T]^{-(\frac{2n+1}{n+1})}, \tag{20}$$

$$V(\phi) = \frac{1}{2}\left[\frac{\phi_0(2n+1)}{\mu^2}\right]^2[(n+1)T]^{\frac{-2(3n+2)}{(n+1)}} - \frac{n(n+2)}{(n+1)^2}\frac{1}{T^2}. \tag{21}$$

The physical quantities that are important in cosmology are proper volume V^3, expansion scalar θ, shear scalar σ^2 and the deceleration parameter q and have the following expressions for the model given by (19):

$$V^3 = \sqrt{-g} = \mu^2[(n+1)T]^{\frac{2n+1}{n+1}}, \tag{22}$$

$$\theta = \frac{1}{3}u^i_{;i} = \frac{2n+1}{3(n+1)^2}\left[\frac{1}{T}\right], \tag{23}$$

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{54}\frac{(2n+1)^2}{(n+1)^4}\left[\frac{1}{T^2}\right], \tag{24}$$

$$q = -3\theta^{-2}\left[\theta_{;i}u^i + \frac{1}{3\theta^2}\right] = \frac{-9n^2(n+1)^2}{(2n+1)^2}. \tag{25}$$

The spatial volume increases with time T when $n + 1 > 0$. Also, when $T \to \infty$, the spatial volume $V \to \infty$. Thus inflation is possible in axially symmetric space-time with a mass less scalar field in the flat region where the potential $V(\phi)$ is constant. Also, the model inflates because of the fact that the deceleration parameter q is negative. The scalar expansion $\theta \to \infty$, when $T \to 0$ and θ becomes finite when $T \to \infty$. The Shear scalar σ tends to infinity for large values of T. The Higg's field ϕ and the potential $V(\phi)$ have initial singularities and for large T, they become zero. Since $\text{Lim}_{T\to\infty}(\sigma/\theta) \neq 0$, the model (19) does not approach isotropy for large T.

5 Conclusions

Axially symmetric cosmological models with self-interacting scalar field are important in the study of early stages of evolution of the universe. Here we have presented an axially symmetric inflationary universe in the presence of mass less scalar field with a flat potential in general relativity Our model is simple and elegant when compared to the inflationary model obtained by Raj Bali and Jain [11]. It is observed that the model is non-singular and does not approach isotropy for large T.

References

1. Burd, A.B., Barrow, J.D.: Nucl. Phys. B **308**, 923 (1988)
2. Wald, R.: Phys. Rev. D **28**, 2118 (1983)
3. Barrow, J.D.: Phys. Lett. B **187**, 12 (1987)
4. Ellis, G.F.R., Madsen, M.S.: Class Quantum Gravity **8**, 667 (1991)
5. Heusler, M.: Phys. Lett. B **253**, 33 (1991)
6. Guth, A.H.: Phys. Rev. D **23**, 347 (1981)
7. Linde, A.D.: Phys. Lett. B **108**, 389 (1982)
8. La, D., Steinharth, P.J.: Phys. Rev. Lett. **62**, 376 (1989)
9. Stein-Schabes, J.A.: Phys. Rev. D **35**, 2345 (1987)
10. Bhattacharjee, R., Baruah, K.K.: Ind. J. Pure Appl. Math. **32**, 47 (2001)
11. Raj, B., Jain, V.C.: Pramana J. Phys. **59**, 1 (2002)
12. Rahaman, F., Bag, G., Bhui, B.C., Das, S.: Fizika B **12**, 193 (2003)
13. Bhattacharya, S., Karade, T.M.: Astrophys. Space Sci. **202**, 69 (1993)
14. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space–Time. Cambridge University Press, Cambridge (1976)